

# ON THE HEEGAARD GENUS OF CONTACT 3-MANIFOLDS

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**ABSTRACT.** It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. We show that Heegaard genus of contact 3-manifolds is not necessarily additive under *contact* connected sum. We also prove some basic properties of the contact genus (a.k.a. open book genus [8]) of 3-manifolds, and compute this invariant for some 3-manifolds.

## 1. INTRODUCTION

We assume that all 3-manifolds are closed, connected and oriented and all contact structures are co-oriented and positive throughout this paper. Let  $Y$  denote a 3-manifold. Given an open book  $(B, \pi)$  on  $Y$ , where  $B$  denotes the binding and  $\pi$  denotes the fibration of  $Y - B$  over  $S^1$ . It follows that  $(\pi^{-1}([0, 1/2]) \cup B)$  and  $(\pi^{-1}([1/2, 1]) \cup B)$  are handlebodies which induce a Heegaard splitting of  $Y$ , where we view  $S^1$  as the interval  $[0, 1]$  whose endpoints are identified with each other. In this sense an open book can be viewed as a special Heegaard splitting. Note that a stabilization of an open book at hand corresponds to a stabilization of the induced Heegaard splitting.

We define the Heegaard genus  $\text{Hg}(Y, \xi)$  of a contact 3-manifold  $(Y, \xi)$  as the minimal genus of a Heegaard surface in any Heegaard splitting of  $Y$  induced from an open book supporting  $\xi$ . Equivalently,  $\text{Hg}(Y, \xi) = 1 + \text{sn}(\xi) = \min\{1 - \chi(\Sigma) \mid \Sigma \text{ is a page of an open book supporting } \xi\}$ , where  $\text{sn}(\xi)$  denotes the support norm of  $\xi$  (cf. [4]) and  $\chi(\Sigma)$  denotes the Euler characteristic of  $\Sigma$ . This is certainly a generalization of the Heegaard genus adapted to contact 3-manifolds. It is well-known that Heegaard genus is additive under connected sum of 3-manifolds. Here we show that Heegaard genus is sub-additive but not necessarily additive under connected sum of *contact* 3-manifolds.

Moreover we define the contact genus  $\text{cg}(Y)$  of a 3-manifold  $Y$  as the minimal Heegaard genus over all contact structures, i.e.,  $\text{cg}(Y) = \min\{\text{Hg}(Y, \xi) \mid \xi \text{ is a contact structure on } Y\}$  which, by Giroux's correspondence [5], is the minimal genus of a Heegaard surface in any Heegaard splitting of  $Y$  induced from an open book. In other words, the contact genus of a 3-manifold is a topological invariant obtained by taking the minimum of the sum  $2g + r - 1$  over all open books, where  $g$  and  $r$  denote the genus of the page and the

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number of binding components of the open book, respectively. We show that contact genus is sub-additive (and conjecture that it is additive) under connected sum of 3-manifolds.

We would like to point out that the contact invariant was first studied by Rubinstein who named it the open book genus of  $Y$  (cf. [8]). We prefer to call it the contact genus to emphasize its connection with contact topology. It is clear by definition that for any contact structure  $\xi$  on  $Y$  we have

$$\text{Hg}(Y) \leq \text{cg}(Y) \leq \text{Hg}(Y, \xi),$$

where  $\text{Hg}(Y)$  denotes the Heegaard genus of  $Y$ . In [8], it was shown that “most” 3-manifolds of Heegaard genus 2 have contact genus  $> 2$ , which implies the existence of 3-manifolds where the first inequality above is strict. In particular, it follows that not every Heegaard splitting of a 3-manifold comes from an open book.

Here we show that “most” 3-manifolds of Heegaard genus 1 have contact genus  $> 1$ . Namely we show that a lens space which is not diffeomorphic to an oriented circle bundle over  $S^2$  have contact genus  $\geq 2$ . On the other hand, the contact genus of any oriented circle bundle over  $S^2$  is equal its Heegaard genus. We also show that there are many small Seifert fibered 3-manifolds (which are not lens spaces) which have this property. Examples of such 3-manifolds were constructed in [8], but our examples are much simpler. We refer the reader to [3] and [7] for more on open books and contact structures.

## 2. HEEGAARD GENUS AND CONTACT CONNECTED SUM

Given any two contact 3-manifolds  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$ . By removing a Darboux ball from each of these contact 3-manifolds and gluing them along their convex boundaries by an orientation reversing map carrying respective characteristic foliations onto each other we get a well defined contact structure  $\xi_1 \# \xi_2$  on the connected sum  $Y_1 \# Y_2$ . The contact 3-manifold  $(Y_1 \# Y_2, \xi_1 \# \xi_2)$  is called the contact connected sum of  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$ . It is well-known that Heegaard genus is additive under connected sum of smooth 3-manifolds, which follows from Haken’s Lemma. Here we show that

**Theorem 1.** *The Heegaard genus is sub-additive but not necessarily additive under connected sum of contact 3-manifolds.*

*Proof.* Let  $\mathcal{OB}_i$  denote the open book realizing  $\text{Hg}(Y_i, \xi_i)$ , for  $i = 1, 2$ . Then the contact structure  $\xi_1 \# \xi_2$  on  $Y_1 \# Y_2$  is supported by the open book  $\mathcal{OB}$  obtained by plumbing the pages of the open books  $\mathcal{OB}_1$  and  $\mathcal{OB}_2$  by Torisu [9]. Denote a page of the open book  $\mathcal{OB}_i$  by  $\Sigma_i$ . It follows that

$$-\chi(\Sigma) = -\chi(\Sigma_1) - \chi(\Sigma_2) + 1,$$

where  $\Sigma$  denotes the page of the open book  $\mathcal{OB}$ . Thus we have

$$\text{Hg}(Y_1 \# Y_2, \xi_1 \# \xi_2) \leq \text{Hg}(Y_1, \xi_1) + \text{Hg}(Y_2, \xi_2),$$

which implies that  $\text{Hg}$  is sub-additive under contact connected sum.

Next we show that  $\text{Hg}$  is not necessarily additive under contact connected sum. Let  $\xi_d$  denote the overtwisted contact structure in  $S^3$  whose  $d_3$  invariant (cf. [6]) is equal to the half integer  $d$ . The following result was obtained in [1]: If  $(Y, \xi)$  is a contact structure with  $c_1(\xi)$  torsion, then

$$d_3(Y, \xi \# \xi_d) = d_3(Y, \xi) + d_3(S^3, \xi_d) + 1/2.$$

Now suppose that  $Y$  is an integral homology sphere. It follows that  $c_1(\xi) = 0$  for every contact structure  $\xi$  on  $Y$ , and  $Y$  carries a unique  $\text{spin}^c$  structure. Thus for an arbitrary contact structure  $\xi$  on  $Y$  we have

$$d_3(Y, \xi \# \xi_{-\frac{1}{2}}) = d_3(Y, \xi) + d_3(S^3, \xi_{-\frac{1}{2}}) + \frac{1}{2} = d_3(Y, \xi),$$

which implies that the connected sum  $\xi \# \xi_{-\frac{1}{2}}$  is homotopic to  $\xi$  as oriented plane fields (cf. [6]). In fact,  $\xi \# \xi_{-\frac{1}{2}}$  is isotopic to  $\xi$  by the classification of overtwisted contact structures due to Eliashberg [2]. As a consequence we have

$$\text{Hg}(Y, \xi \# \xi_{-\frac{1}{2}}) = \text{Hg}(Y, \xi).$$

On the other hand, in ([4], Lemma 5.5), it was proved that  $\text{Hg}(S^3, \xi_{-\frac{1}{2}}) = 2$ . Note that an open book realizing  $\text{Hg}(S^3, \xi_{-\frac{1}{2}})$  can be described by taking a pair of pants as a page and  $t_1 t_2^{-2} t_3^{-3}$  as the monodromy, where  $t_i$  denotes a right-handed Dehn twist along a boundary component. Consequently we have

$$\text{Hg}(Y \# S^3, \xi \# \xi_{-\frac{1}{2}}) < \text{Hg}(Y, \xi) + \text{Hg}(S^3, \xi_{-\frac{1}{2}}).$$

□

### 3. CONTACT GENUS OF THREE DIMENSIONAL MANIFOLDS

Here we provide some basic properties of the contact genus of 3-manifolds, and compute this invariant for some 3-manifolds.

**Proposition 2.** *Let  $Y$  denote a 3-manifold. Then we have*

- (a)  $\text{cg}(Y) \geq 0$  ( $= 0$  if and only if  $Y \cong S^3$ ),
- (b)  $\text{cg}(Y) = 1$  if and only if  $Y$  is an oriented circle bundle over  $S^2$  (which is not diffeomorphic to  $S^3$ ).

*Proof.* For a 3-manifold  $Y$ ,  $\text{cg}(Y)$  is obtained by taking the minimum of the sum  $2g + r - 1$  over all open books, where  $g$  and  $r$  denote the genus of the page and the number of binding components of an open book, respectively. Hence we have  $0 \leq \text{cg}(Y)$  for an arbitrary 3-manifold  $Y$ , since  $g \geq 0$  and  $r \geq 1$ . It is clear that the absolute minimum of the expression  $2g + r - 1$  is realized when  $g = 0$  and  $r = 1$  and the open book with disk pages and trivial monodromy supports the unique tight contact structure on  $S^3$ , which proves (a).

To prove (b), we note that  $\text{cg}(Y) = 1$  is realized if and only if  $g = 0$  and  $r = 2$ . Any self-diffeomorphism of an annulus is given by  $t_c^m$ , for some  $m \in \mathbb{Z}$ , where  $c$  is the core of the annulus, and  $t_c$  denotes a right-handed Dehn twist along  $c$ . If  $m \geq 0$ , this open book supports the unique tight contact structure on the lens space  $L(m, -1)$  which is an oriented circle bundle over  $S^2$  with Euler number  $m$ . Otherwise (i.e., when  $m < 0$ ) the induced contact structure is the overtwisted contact structure on  $L(-m, 1)$  which is an oriented circle bundle over  $S^2$  with Euler number  $m$ . Combining, we showed that  $\text{cg}(Y) = 1$  if and only if  $Y$  is an oriented circle bundle over  $S^2$ , which is not diffeomorphic to  $S^3$ .  $\square$

Note that oriented circle bundles over  $S^2$  are very special lens spaces and therefore we immediately conclude from Proposition 2 that

**Corollary 3.** *Most 3-manifolds of Heegaard genus 1 have contact genus  $> 1$ .*

For example,  $\text{cg}(L(5, 3)) = 2$ , since  $L(5, 3)$  is not a circle bundle over  $S^2$  and it carries a (tight) contact structure which is supported by a planar open book with three binding components.

**Lemma 4.** *We have  $\text{cg}(Y_{p,q,r}) \leq 2$ , where  $Y_{p,q,r}$  denotes the 3-manifold depicted in Figure 1, with  $p, q, r \in \mathbb{Z}$ . Moreover if  $|p| > 1$ ,  $|q| > 1$  and  $|r| > 1$  then  $\text{cg}(Y_{p,q,r}) = 2$ .*

*Proof.* It follows from [4] that  $Y_{p,q,r}$  has a planar open book with at most three binding components, which indeed proves that  $\text{cg}(Y_{p,q,r}) \leq 2$ . Moreover, under the assumption that  $|p| > 1$ ,  $|q| > 1$ , and  $|r| > 1$ , the 3-manifold  $Y_{p,q,r}$  is not diffeomorphic to any lens space and hence  $\text{cg}(Y_{p,q,r}) = 2$  by Proposition 2.

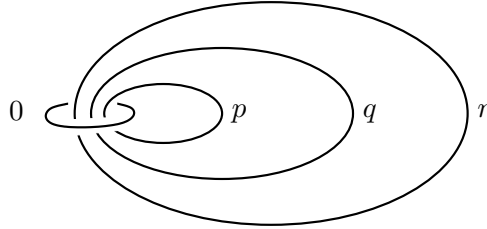


FIGURE 1. Integral surgery diagram for the small Seifert fibered 3-manifold  $Y_{p,q,r}$

$\square$

When we drop the assumption on  $p, q$  and  $r$  in Lemma 4, we observe that  $Y_{p,q,r}$  is diffeomorphic to either  $S^3$ ,  $S^1 \times S^2$ , a lens space, or certain connected sums of these for some values of the integers  $p, q$  and  $r$ .

**Remark 5.** Note that Lemma 4 exhibits examples of 3-manifolds  $Y = Y_{p,q,r}$  for which  $\text{Hg}(Y) = \text{cg}(Y) = 2$ , although most 3-manifolds of Heegaard genus 2 have contact genus  $> 2$  as was shown by Rubinstein [8].

**Lemma 6.** We have  $\text{cg}(\#_k S^1 \times S^2) = k$ , for  $k \geq 1$ .

*Proof.* Since  $\text{Hg}(\#_k S^1 \times S^2) = k$ , we know that  $\text{cg}(\#_k S^1 \times S^2) \geq k$ . Hence to show that  $\text{cg}(\#_k S^1 \times S^2) = k$ , we just need to realize this lower bound by a Heegaard splitting of  $\#_k S^1 \times S^2$  induced from an open book. We use the fact that the unique tight contact structure on  $\#_k S^1 \times S^2$  is supported by an planar open book with  $k+1$  binding components, whose monodromy is the identity map. □

The proof of the following result is similar to the proof of Theorem 1.

**Proposition 7.** Let  $Y_i$  denote a 3-manifold, for  $i = 1, 2$ . Then we have

$$\text{cg}(Y_1 \# Y_2) \leq \text{cg}(Y_1) + \text{cg}(Y_2).$$

**Conjecture 8.** Contact genus is additive under connected sum of 3-manifolds.

Note that if  $\text{Hg}(Y_i) = \text{cg}(Y_i)$  for  $i = 1, 2$ , then  $\text{cg}(Y_1 \# Y_2) = \text{cg}(Y_1) + \text{cg}(Y_2)$ .

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